absorbing extra heat flux and hence eliminates the temperature distortion at the cavity base during the transient period. To illustrate this fact we examine Fig. 3 for the data $\kappa_3/\kappa_l = 1$ and $\rho_3 c_3/\rho_l c_l = 1.3$. One sees that when $d_t/d = 1$ the temperature distortion can indeed be negative. Therefore, if d_t/d is chosen between 0.8 and 1 the error can be minimized. However one must keep in mind that the elimination of error by heat capacitance can work during the transient period only, for once steady-state conduction is established the heat capacity ρc will no longer have any affect, and overheating at the cavity base eventually will develop. This can be best seen from the governing equation, Eq. (1), where for steady state the unsteady term which contains the ρc product is zero and is not a parameter affecting the distortion.

Another important fact that should be mentioned is that, in general, the optimum choice of d_t/d ratio for a given κ_3/κ_l and ρc ratio is rather insensitive to the variation of the ϵ/D ratio ranging from 0.02 to 0.1. This fact was already pointed out by Chen and Danh[9] in their experiment which demonstrated that the temperature distortion at the base of the cavity is more sensitive to the variation of the cavity diameter than the depth of the cavity drilled.

One disadvantage of invoking finite element analysis is that the result does not give a clear functional relation among the parameters involved. In an attempt to obtain a simple and approximate relation to relate the various parameters we note the following results: 1) the optimum d_t/d ratio for zero temperature distortion is a strong function of κ_3/κ_l and ρc ratio but is relatively insensitive to the ϵ/D ratio, and 2) from theoretical reasoning the d_t/d ratio is independent of ρc ratio at steady state. A simple steady-state one-dimensional analysis in which the thermocouple and the insulation material in the cavity is made to conduct the same amount of heat that would be transferred without the cavity gives the relation

$$d_t/d = \sqrt{(\kappa_l - \alpha_2)/(\kappa_3 - \kappa_2)} \tag{5}$$

Using the above equation as a base we find that for transient heat conduction as calculated by the finite element method, Eq. (6) correlates very well with the optimum d_t/d ratio:

$$d_t/d = (\rho_3 c_3/\rho_l c_l)^{0.3} \sqrt{(\kappa_l - \kappa_2)/(\kappa_3 - \kappa_2)} \tag{6}$$

Equation (6) gives a distortion error of no more than two percentage points. In practice, Eq. (6) may be used as a rule of thumb.

In summary, the temperature distortion caused by a cavity drilled into a disk to accommodate a thermocouple has been studied. The calculation is carried out for the case of constant heat flux. It is shown that the difference in temperature at the base of the cavity from that without a cavity can be eliminated by a properly chosen combination of the ratio of the thermocouple diameter to the cavity diameter, d_t/d, and the thermocouple material. The optimum ratio of d_t/d can be found from Fig. 3 or approximately from Eq. (6). As a rule, the thermocouple material must be chosen so that it has a higher thermal conductivity than that of the heat conducting solid. The cavity diameter should be as small as practically possible.

Acknowledgment

This work was supported by ARO research grant DAA-G29-76-G-0123.

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Nonlinear Flap-Lag-Axial Equations of a Rotating Beam

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Introduction

THE literature dealing with the dynamics of rotating elastic bodies such as spin-stabilized satellites, helicopter rotor blades, and whirling beams has proliferated in the last decade. Representative papers treating the dynamic aspects of such structures are given in Refs. 1 to 11. Examining these references, one may identify essentially four approaches by which analysts have proceeded to establish either the linear or nonlinear governing equations of motion for the particular problem addressed. These are: the effective applied load artifice[2,3] in combination with a variational principle; the use of Newton's second law, written as D'Alembert's principle, applied to the deformed configuration;[1,7,10] a variational approach in which nonlinear strain-displacement relations and a first-degree displacement field are used;[5,9,10] and the method introduced by Vigneron[11] for deriving the linear flap-lag equations of a rotating beam. While these approaches complement one another, their application in some of the cited references reveals confusion regarding several aspects of the development of both the linear and nonlinear equations of motion of a rotating blade or beam. The confusion centers about: whether the geometric nonlinear theory of elasticity has been (or must be) used in deriving the equations of motion; whether Houbolt and Brooks[1] have considered geometric nonlinearity and thus have the necessary ingredients for extending their development to obtain the second-degree nonlinear equations; whether or not foreshortening must be considered explicitly by including it in the assumed axial displacement field; the use of the inex-

Received Dec. 10, 1976; revision received March 29, 1977.

Index categories: Structural Dynamics; Propeller and Rotor Systems.

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tensibility assumption and its interpretation in developing the equations of motion.

The purposes of this Note are to: show that the four approaches identified above all make use of the geometric nonlinear theory of elasticity either implicitly or explicitly; introduce an alternative approach for deriving the nonlinear coupled flap-lag-axial equations of motion of a rotating beam; offer some clarifying comments on the role of foreshortening and the use of the inextensibility assumption in these various approaches.

Comparison of Methods

In this section the four approaches for developing the equations of motion for a rotating beam are compared and discussed with a view toward bringing the approaches into perspective and demonstrating that they all use, either implicitly or explicitly, the geometric nonlinear theory of elasticity, an ingredient which is necessary to obtain even the correct linear equations. Geometric nonlinearity[12,13] is associated with the necessity to consider the deformed configuration to write the equilibrium equations and to include nonlinear terms in the strain-displacement relations. The need to consider geometric nonlinearity in linear formulations has not been generally recognized in the literature. For example, Jones and Bhuta[14] developed the linear flap-lag-axial equations of a rotating beam using the classical (i.e., linear) theory of elasticity and were unable to recover the linear terms associated with geometric nonlinearity.

In the effective applied load method, which is the familiar textbook method for determining the linear vertical bending equation of a rotating beam,[2,3] geometric nonlinearity is reflected in the expression for the work done by the equilibrium centrifugal force as the beam foreshortens axially. This fact has only recently been clearly brought out in the literature.[6,9] Specifically, second-degree foreshortening terms due to bending are implicitly included in the axial displacement field when calculating the work done by the centrifugal axial force. It is interesting to note, however, that in this textbook approach these second-degree terms are not addressed while calculating the kinetic energy.

The second approach identified earlier is based on the use of Newton's second law applied to the deformed configuration.[1,7,10] Using this approach, Houbolt and Brooks[1] derived a system of linear equations of motion. Friedmann,[7] in one extension of their work to the nonlinear case, included a second-degree term in the assumed axial displacement field to account for the axial foreshortening due to the change of length caused by the transverse bending displacements. Hodges and Dowell,[10] in another extension of Ref. 1 to the nonlinear case, used a nonlinear expression for the axial strain component but did not include axial foreshortening due to bending in their assumed displacement field. It should be noted that Houbolt and Brooks[1] wrote their equilibrium equations for an element of a beam in the deformed configuration. This implies that allowance was made for arbitrary rotations and indicates that the nonlinear strain-displacement relations given in Eqs. 1.119 of Novozhilov[13] are employed, albeit implicitly, in their derivation. Hence, they have accounted for the geometric nonlinear theory of elasticity to the extent necessary to recover all the first-degree (i.e., linear) terms in the governing equations. This implicit consideration of the geometric nonlinear theory of elasticity by Houbolt and Brooks has apparently not been recognized (for example, see Ref. 5).

The third approach identified can be used to obtain both the linear and nonlinear equations of a rotating beam. However, in order to obtain the second-degree nonlinear equations, terms through fourth degree must be retained in the energy expressions.‡ References 5 and 10 both employed this approach. It should be noted that the third approach is the one usually employed for a general three-dimensional rotating body.[6,9]

The fourth approach identified was introduced by Vigneron[11] in commenting on a paper by Likens, et al.[6] His approach is characterized by the use of the nonlinear strain-displacement relations and an axial displacement field which includes the second-degree foreshortening terms due to bending. To obtain the linear flap-lag equations of motion for a rotating beam, Ref. 11 retained terms through second degree in the kinetic and potential energy expressions. As a result of using foreshortening explicitly, the centrifugal stiffening terms enter the final equations through the kinetic energy rather than through the potential energy as in the first and third approaches. In the next section this fourth approach will be extended to derive the second-degree nonlinear flap-lag-axial equations of motion of a rotating beam.

‡The kinetic energy turns out to have no third or fourth-degree terms.

Alternative Method for Deriving the Nonlinear Equations

The procedure of Ref. 11 is extended in this section to derive the second-degree nonlinear flap-lag-axial vibration equations of motion by retaining terms through third degree in both the kinetic and potential energy expressions. The extension to the nonlinear case by this approach is new. Also, it should be remarked that the approach to be given parallels the development of the equations of a nonrotating beam by a variational method wherein retention of second-degree nonlinear terms in the kinetic and potential energy expressions leads to linear equations, retention of third-degree terms to second-degree nonlinear equations, and so forth.

Consider a beam or a rotor blade with zero pitch and twist and no precone.§ The elastic axis before deformation is assumed to be aligned with the x axis of an orthogonal rotating triad xyz with origin at one end of the beam. The z axis is taken positive upward and the y axis positive forward to complete the right-handed coordinate system. The beam rotates about the z axis with constant angular velocity Ω. Let u, v, and w be the elastic deformations of the elastic axis in the directions x,y, and z, respectively. For the particular case in which shear deformation and rotary inertia are taken to be zero, the position vector of an arbitrary point in the cross section of the beam after deformation can be approximated by

$$x_I = x + u_I = u - yv' - zw' - U_F \tag{1a}$$

$$y_I = y + v_I = y + v \tag{1b}$$

$$z_I = z + w_I = z + w \tag{1c}$$

where U_F, the axial displacement associated with the foreshortening of the elastic axis due to bending, is given by

$$U_F = 1/2 \int_0^x (v'^2 + w'^2)\,\mathrm{d}x \tag{2}$$

Following usual practice in solid mechanics, Green's strain tensor is employed to express the strains in terms of displacements. Ignoring all strains other than that in the axial direction, the axial component of Green's strain tensor (from Ref. 13) is given by

$$\epsilon_{xx} = u_I' + 1/2[u_I'^2 + v_I'^2 + w_I'^2] \tag{3}$$

Substituting u_I, v_I, and w_I from Eq. (1) into Eq. (3) and retaining terms through second degree in the dependent

§These assumptions have been made for simplicity. However, the inclusion of these parameters is straightforward and, in fact, is under consideration by the authors.

variables u, v, and w leads to

$$\epsilon_{xx}=u'-yv''-zw''+1/2(u'-yv''-zw'')^2 \tag{4}$$

For the case of small strains in which the elongations and shears are negligible compared to unity, the axial component of Green's strain tensor, ϵ_{xx}, is identical to the well-known engineering strain (called relative elongation in Ref. 13). Invoking the small strain assumption, then using Hooke's law, the strain energy through third degree in the dependent variables is given by

$$V=\frac{E}{2}\int_0^R (Au'^2+I_{zz}v''^2+I_{yy}w''^2)\,dx + \frac{E}{2}\iiint(u'-yv''-zw'')^2 dxdydz \tag{5}$$

where

$$A=\iint dydz \qquad I_{zz}=\iint y^2 dydz \qquad I_{yy}=\iint z^2 dydz$$

and it is assumed that

$$\iint y dydz=\iint z dydz=\iint yz dydz=0 \tag{6}$$

The corresponding kinetic energy to third degree assumes the form

$$T=1/2\int_0^R \{m(\dot u^2+\dot v^2+\dot w^2)-2m\dot u\,\dot U_F\}dx + \frac{\Omega^2}{2}\int_0^R \{m(x^2+u^2+v^2)+2mxu-2mxU_F-2muU_F\}dx - \Omega\int_0^R \{m\dot u v-mv\dot U_F-mx\dot v-m\dot v u+m\dot v U_F\}dx \tag{7}$$

where m is the mass per unit length. Assuming uniform mass and section properties and using Eqs. (5) and (7) in Hamilton's principle leads to

$$m(\ddot u-\ddot U_F-\Omega^2 u+\Omega^2 U_F-\Omega^2 x-2\dot v\Omega) -AE(u''+\underline{3u'\,u''})\underline{-3EI_{zz}v''v'''}\ \underline{-3EI_{yy}w''w'''}=0 \tag{8a}$$

$$m(\ddot v-\Omega^2 v+2\Omega\dot u-2\Omega\dot U_F)-(Tv')' +EI_{zz}v''''\underline{+3EI_{zz}(2u''v'''+u'v''''+v''u''')}=0 \tag{8b}$$

$$m\ddot w-(Tw')'+EI_{yy}w'''' \underline{+3EI_{yy}(2u''w'''+u'w''''+w''u''')}=0 \tag{8c}$$

where

$$T=m\int_x^R (-\ddot u+\Omega^2 x+\Omega^2 u+2\Omega\dot v)dx \tag{8d}$$

The underlined terms in Eqs. (8) are a consequence of retaining the term $1/2\ u_l'^2$ in Eq. (3). Alternatively, without considering foreshortening, the equations of motion given by Eq. (8) including the underlined terms can also be obtained by using the third approach described earlier if the $u_l'^2$ term in Eq. (3) is retained. As mentioned earlier, the retention of the term $1/2\ u_l'^2$ in the axial nonlinear strain-displacement relation is consistent with the definition of Green's strain tensor. However, $u_l'^2$ is oftentimes discarded a priori on the basis of an ordering scheme. It should also be noted that if the sectional rotations are assumed small compared to unity, then the term $u_l'^2$ can be neglected (Ref. 13, page 52). With this assumption, the underlined terms in Eqs. (8) will not appear. It is interesting to note that this term does not appear in Eq. (3) if the small strain assumption is not invoked and one retains terms through second degree in u_l, v_l, and w_l in the expansion of the expression for the engineering strain (elongation) E_l given by

$$E_l=\sqrt{1+2\epsilon_{xx}}-1 \tag{9}$$

Thus, if one uses E_l for the axial strain in place of ϵ_{xx}, the underlined terms in Eqs. (8) will not appear. However, in this case the engineering strain E_l is not equal to ϵ_{xx}. To compare the resulting equations with those in the cited references the underlined terms in Eqs. (8) will not be addressed further herein. However, the implications of retaining the underlined terms are being investigated.

Assuming the beam to be inextensible (i.e., u and all its derivatives are zero) equations (8) can be reduced to the second-degree nonlinear flap-lag equations arrived at by Friedmann[7] and Hodges and Ormiston.[5] Furthermore, a careful examination of the development of Houbolt and Brooks reveals that their derivation contains the ingredients necessary for extending their development to obtain the second-degree nonlinear flap-lag-axial equations of motion. To this end one would merely have to consider the nonlinear axial strain-displacement relation by modifying their Eq. (A12) specialized to the case of flap-lag-axial to read, in their notation,

$$\epsilon=u'\underline{+1/2(v'^2+w'^2)}-\eta(v''\cos\beta+w''\sin\beta) -\zeta(w''\cos\beta-v''\sin\beta) \tag{10}$$

where the additional (nonlinear) terms are underlined. If this equation is solved for u on the elastic axis and substituted into their Eq. (B11), the resulting second-degree equations of motion agree directly with those of Friedmann,[7] and with those of Hodges and Ormiston[5] if the same elimination of $\dot u$ is carried out in Ref. 5 using Eqs. (21a to 21d). These observations are believed to be new. For descriptive purposes, the extension of the Houbolt and Brooks approach just described is designated the extended Houbolt and Brooks approach herein.

Foreshortening

The question as to whether or not foreshortening must be explicitly considered by including terms through second degree in the assumed axial displacement field is a moot one. From physical considerations, foreshortening results from the changes in length caused by the transverse bending displacements. Also, experience gained in deriving even the linear flapping equations of motion of a rigid blade hinged at the center of rotation indicates that one must retain all second-degree terms in any absolute velocity components which have zeroth-degree terms associated with the angular velocity of rotation. These second-degree terms stem from second-degree terms in the axial component of the position vector of a blade mass element. The retention of these terms is necessary to insure that all second-degree terms in the kinetic energy are obtained. For an elastic blade the equivalent second-degree terms in the axial displacement field are precisely the foreshortening terms. Based on this similarity it appears that the second-degree foreshortening terms must be included in the axial displacement field. This is the approach used by Vigneron for deriving the linear flap-lag equations. The inclusion of the second-degree foreshortening terms in the axial displacement field when deriving linear equations of motion may appear to be inconsistent. However, these terms are a reflection of the geometric nonlinearity which must be employed for establishing even the linear equations of motion of a rotating beam or rotor blade.

The foreshortening effect may also be accounted for without explicitly including the second-degree terms in the axial displacement field. Special considerations are required in this case, however. Specifically, when proceeding by an approach based on a variational method, terms through fourth degree must be retained in the energy expressions and explicit use must be made of the nonlinear strain-displacement relations [such as that given by Eq. (3) for the axial strain] to develop both the linear and second-degree nonlinear equations. In this case one cannot make the inextensibility assumption either a priori or at an early stage in the development (This point will be clarified in the next section). Such an approach was employed by Likens et al.[6] for linear equations and Hodges and Ormiston[5] for nonlinear equations. In Ref. 5, for example, one has to eliminate the coriolis term involving $\dot{u}$ from the lag equation using the nonlinear strain-displacement relation given in Eq. (21d). One can also elect to use the Newtonian approach in combination with D'Alembert's principle to develop either the linear or second-degree nonlinear equations of motion without explicitly considering foreshortening. Houbolt and Brooks[1] did this for linear equations and Hodges and Dowell[10] for nonlinear equations. As already stated, the procedure of Houbolt and Brooks can be extended to develop the second-degree nonlinear flap-lag-axial equations by explicitly using the nonlinear strain-displacement relation given in Eq. (3) above (with the first term in the brackets discarded) to eliminate $\dot{u}$ from their lag equation.

In summary, in developing either the linear or nonlinear equations of motion for a rotating beam or rotor blade by either a variational or Newtonian procedure the analyst has the option of either explicitly or implicitly considering foreshortening. The approach with the explicit inclusion of foreshortening in the axial displacement field parallels the familiar steps in deriving the corresponding equations of a nonrotating beam by a variational method wherein retention of second-degree terms in the kinetic and potential energy expressions leads to linear equations, the retention of third-degree terms to second-degree nonlinear equations, and so forth. The other approach requires special considerations.

Inextensibility

An expedient which is usually employed during the development of the equations of motion for a rotating beam or a helicopter rotor blade in order to eliminate the axial equation of motion is that of "inextensibility." According to Vigneron[11] and the present authors, inextensibility is taken to mean that the axial extension of the elastic axis, u, and all its derivatives are zero. This particular definition implies[15] that the beam is axially rigid, that is, that the axial rigidity AE is infinite. If one chooses to impose the inextensibility assumption a priori, foreshortening must be explicitly considered by including it in the assumed axial displacement field, otherwise one will not obtain the proper second-degree nonlinear equations of motion. Such an incorrect approach was used by Dow.[8] This fact can also be verified by setting $\dot{u}$ equal to zero in Eq. (21b) of Ref. 5 or in the corresponding equations of Houbolt and Brooks as extended earlier in this Note. It should be noted, however, that if one chooses not to consider foreshortening explicitly and imposes the inextensibility assumption after establishing the final equations of motion the correct equations will be obtained. Such a procedure was employed by Hodges and Ormiston.[5] While using this approach, these authors assumed that the beam was inextensible in terms of "perturbation" deflections. However, if $\dot{u}$ is eliminated from their Eq. (21b) using Eq. (21d), it can be shown that their assumption that the axial rigidity parameter K is large is equivalent to saying that beam is axially inextensible in terms of both steady-state and perturbation deflections.

In summary, the inextensibility assumption can be made either a priori or at any stage in the development of the equations of motion if foreshortening is explicitly considered but must be made a posteriori if foreshortening is implicitly considered.

Conclusions

All methods which are employed for deriving even the linear flap-lag equations of motion of a rotating beam or a helicopter rotor blade use the geometric nonlinear theory of elasticity, at least implicitly. This fact has not been generally recognized in the literature.

An alternative approach for deriving the second-degree nonlinear flap-lag-axial equations of motion for a rotating beam has been presented. This approach is appealing in that it parallels the corresponding development of the equations for a nonrotating beam where retention of second-degree terms in the kinetic and potential energy expressions leads to linear equations, retention of third-degree terms to second-degree nonlinear equations, and so forth.

The retention of the term $1/2\ u_l'^2$ in the nonlinear axial strain-displacement expression is consistent with the definition of Green's strain tensor and leads to several additional second-degree terms in the flap-lag-axial equations of motion.

Several aspects regarding the use of foreshortening and the inextensibility assumptions in the development of the nonlinear flap-lag-axial equations of motion have been examined.

Using the nonlinear axial strain relation $\epsilon_{xx} = u_l' + 1/2\ (v_l'^2 + w_l'^2)$ in the development of Houbolt and Brooks to eliminate $\dot{u}$ from their lag equation, the nonlinear flap-lag equations of Friedmann, Hodges and Ormiston, and the present authors (after assuming inextensibility) can be recovered for the special case of flap-lag with zero pitch and twist and no precone. This is a new observation.

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